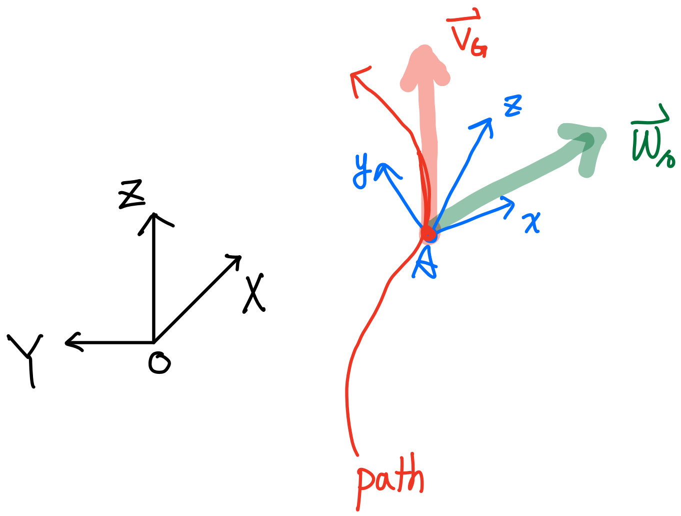
Govern Equations of 6 dof Rigid Body:

$$\vec{F}_G = m(\dot{\vec{V}}_G + \vec{W}_{/o} \times \vec{V}_G) \qquad (1)$$

$$\vec{M}_G = [I_G] \cdot \dot{\vec{W}}_{/o} + \vec{W}_{/o} \times ([I_G] \cdot \vec{W}_{/o}) \qquad (2)$$



* All the vectors are represented in the body-fixed coordinate $A_{xyz}(\hat{i}, \hat{j}, \hat{k})$
* $\vec{W}_{/o}$ is the rotating rate of the body-fixed coordinate A_{xyz} with respect to the inertial coordinate OXYZ.
* $\dot{\vec{V}}_G \triangleq \dot{V}_{Gx}\,\hat{i} + \dot{V}_{Gy}\,\hat{j} + \dot{V}_{Gz}\,\hat{z} = \left(\frac{\partial \vec{V}}{\partial t}\right)_{\vec{w}=0}$

How do we get eq. (1) and (2) ?

From "the time derivative of a vector in a rotating coordinate"

Suppose $\vec{r}$ is a vector represented by a rotating coordinate.

Then, $\frac{d\vec{r}}{dt} = \left(\frac{\partial \vec{r}}{\partial t}\right)_{\text{when } \vec{w}=0} + \vec{W} \times \vec{r}$

So, Newton's Second Law : $\vec{F} = m \cdot \frac{d\vec{V}}{dt}$

If $\vec{V}$ is represented in a rotating coordinate, then

$$\vec{F} = m \cdot \frac{d\vec{V_G}}{dt} = m \cdot \left(\frac{\partial \vec{V_G}}{\partial t}\right)_{\vec{\omega}=0} + \vec{\omega_0} \times \vec{V_G}$$

$$= m \cdot \dot{\vec{V_G}} + \vec{\omega_0} \times \vec{V_G}$$

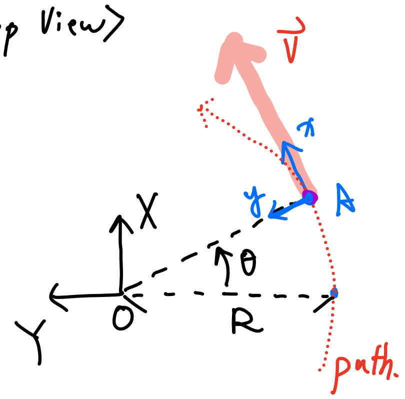
Euler Torque Equation : $\vec{M} = \frac{d(\vec{H})}{dt}$, $\quad \vec{H} = [I_G]\,\vec{\omega_0}$

If $\vec{\omega_0}$ is represented in a rotating coordinate, then.

$$\vec{M} = \frac{d([I]\vec{\omega})}{dt} = [I] \cdot \dot{\vec{\omega}} + \vec{\omega} \times \left([I]\vec{\omega}\right)$$

Example 1: Spacecraft in circular motion with constant Radius.

<Top View>



1) Describe the motion.

$\vec{V} = V_x \hat{i}$

$(\hat{i}, \hat{j}, \hat{k})$ constructs a rotating coordinate A_{xyz} centered on the CM of the body and rotates with respect to O_{XYZ} by $\omega \hat{k}$

2) Use the Force eq. (1)

$$\vec{F} = m\left(\frac{\partial \vec{V}}{\partial t}_A + \vec{\omega} \times \vec{V}\right) = m \cdot (\dot{V_x} \hat{i}) + m \cdot (\omega \hat{k} \times V_x \hat{i})$$

$$= m \cdot \dot{V_x} \hat{i} + m \omega V_x \hat{j}$$

Since $V_x = R \cdot \omega$, $\quad \vec{F} = \underbrace{mR\dot{\omega}}_{①} \hat{i} + \underbrace{mR\omega^2}_{②} \hat{j}$

①: force to accelerate the speed.

②: centrifugal force to constrain the body in a circular path.

3) : Use Torque Eq (2)

$$\vec{M} = [I]\dot{\vec{w}} + \vec{w} \times ([I]\vec{w})$$

$$= \begin{bmatrix} I_x & & \\ & I_y & \\ & & I_z \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{w} \end{bmatrix} + w\hat{k} \times \left(\begin{bmatrix} I_x & & \\ & I_y & \\ & & I_z \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix} \right)$$

$$= I_{zz}\dot{w}\,\hat{k} + w\hat{k} \times (I_{zz} \cdot w\,\hat{k})$$

$$= \underbrace{I_{zz}\dot{w}}_{③}\,\hat{k}$$

③ : As the body rotates faster, w will increase, thus a torque along z axis is required to increase the rotating speed.

Conclusion :

For a body in a circular motion with constant angular acceleration. [R = constant, $\dot{w}$ = constant],

A constant Torque : $M_z = I_{zz}\dot{w}$ is required.

A constant Force tangential to its motion, $F_x = mR\dot{w}$ is required

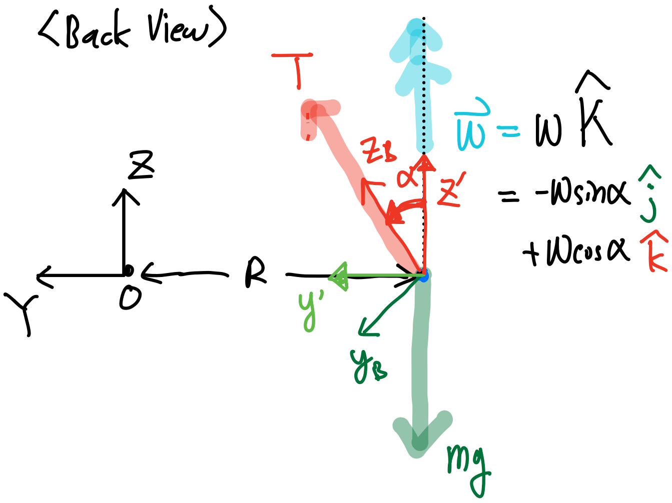
and An INCREASING Centrifugal force, $F_y = mRw^2$ is required, (where $w = w(t) = w(0) + \dot{w} \cdot dt$)

Example 2: Quadcopter circular motion (R, w, $Rw = V_x$ are all constants)

A quadcopter is different from a spacecraft
a) it needs to fight against gravity.
b) its force points only in the z direction.

Thus, for a quadcopter to move in a circular motion, it needs an roll angle α to provide the centrifugal force.

<Back View>



Using the results from example 1, the force required for a body to move in circular path is

$$\left[F_{x'} = mR\cancel{\dot{w}}^{\,0},\ F_{y'} = mRw^2,\ F_{z'} = 0 \right]$$

Thus $\vec{F}^{\,xyz'} = m\vec{a}^{\,xyz'}$ becomes:

$$\begin{cases} y': & T\cdot\sin\alpha\,\hat{j}' = mRw^2\,\hat{j}' \\ z': & (T\cdot\cos\alpha - mg)\,\hat{k}' = 0 \end{cases} \Rightarrow \boxed{\begin{aligned} \alpha &= \tan^{-1}\left(\frac{Rw^2}{g}\right) \\ T &= \frac{mg}{\cos\alpha} \end{aligned}}$$

* NOTE: Here we sum the force along the xyz' frame, because the acceleration is much simple along this frame. xyz' frame rotates with the xyz_{Body} frame. Put its $\alpha = 0$.

Now, let's look at the Torque Eq:

* Note: $w\hat{k} = w(-\sin\alpha\,\hat{j} + \cos\alpha\,\hat{k}) \triangleq w_y\,\hat{j} + w_z\,\hat{k}$

$$\vec{M} = [I]\cancel{\dot{\vec{w}}}^{\,0} + \vec{w}\times([I]\vec{w})$$

$$= w\hat{K}\times\left(\begin{bmatrix} I_x & & \\ & I_y & \\ & & I_z \end{bmatrix}\cdot w\hat{K}\right) = (w_y\hat{j} + w_z\hat{k})\times\left(\begin{bmatrix} I_x & & \\ & I_y & \\ & & I_z \end{bmatrix}\begin{bmatrix} 0 \\ w_y \\ w_z \end{bmatrix}\right)$$

$$= (w_y\hat{j} + w_z\hat{k})\times(I_y w_y\,\hat{j} + I_z w_z\,\hat{k})$$

$$= I_z w_y w_z\,\hat{i} - I_y w_y w_z\,\hat{i}$$

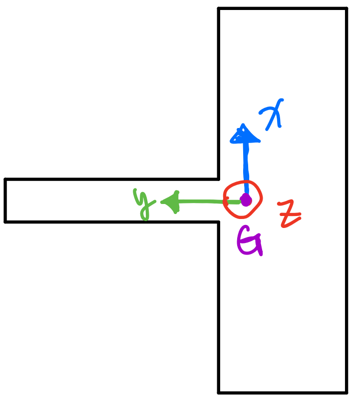
$$= w_y w_z (I_z - I_y)\,\hat{i} = w^2\sin\alpha\cdot\cos\alpha\,(I_z - I_y)\,\hat{i}$$

$$\boxed{M_x = w^2\sin\alpha\cdot\cos\alpha\cdot(I_z - I_y)}$$

So, if Izz is not equal to Iyy, then a constant torque along x-axis is required. This torque is required to maintain the quadcopter's roll angle, which in turns provide the centrifugal force for circular motion.
(Normally Izz is not equal to Iyy for quadcopter.)

Now we are prepared to understand the counter-intuitive rotating behavior of Dzhanibekov effect (or intermediate axis theorem)

For a t-shaped object, Ixx<Iyy<Izz.

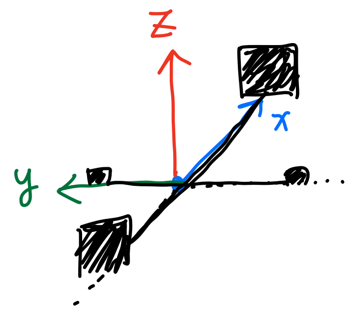


Imagine this moment of inertia ($I_{xx} < I_{yy} < I_{zz}$) is constructed by four mass blocks at the following positions so that we can analyze the rigid body as different point mass.

Point mass dynamics:

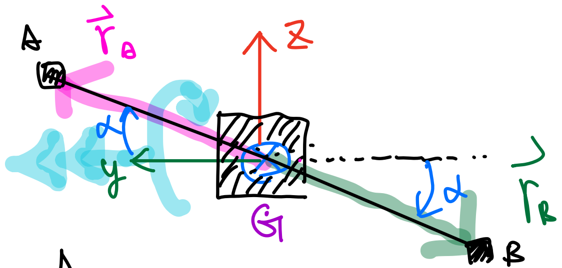
$$\vec{M}_{B/G} = \frac{d}{dt}\left(\vec{h}_{B/G}\right)$$

$\vec{h}$: angular momentum of point B with respect to G.



Now, let's spin the rigid body along its y-axis (the intermediate axis) with a small roll angle along x-axis.

< Back View >



Mass A.

$$\vec{r}_A = r_y\hat{j} + r_z\hat{k}, \quad \vec{V}_B = \omega\hat{j}\times(\vec{r}_A) = \omega\hat{j}\times(r_y\hat{j}+r_z\hat{k})$$

$$= \omega r_z\hat{i}$$

$$\vec{P} = m\vec{V}_B = m\omega r_z\hat{i}$$

$$\vec{h} = \vec{r}_A\times\vec{P} = (r_y\hat{j}+r_z\hat{k})\times(m\omega r_z\hat{i}) = -m\omega r_y r_z\hat{k} + m\omega r_z^2\hat{j}$$

$$\underset{\substack{\text{due}\\\text{to}\\\text{mass } A}}{\vec{M}_{/G}} = \frac{d\vec{h}}{dt} = \vec{w}\times\vec{h} = w\hat{j}\times\left(-m w r_y r_z \hat{k} + \cancel{m w r_z^2 \hat{j}}\right)$$

$$= -m w^2 r_y r_z \hat{i}$$

Mass B :

$$\vec{r}_B = -r_y\hat{j} - r_z\hat{k}, \quad \vec{V}_B = w\hat{j}\times(-r_y\hat{j} - r_z\hat{k}) = -w r_z\hat{i}, \quad \vec{P}_B = -m w r_z\hat{i}$$

$$\vec{h} = \vec{r}_B \times \vec{P}_B = (-r_y\hat{j} - r_z\hat{k})\times(-m w r_z\hat{i}) = -m w r_y r_z\hat{k} + m w r_z^2\hat{j}$$

$$\underset{\substack{\text{due to}\\\text{mass } B}}{\vec{M}_{/G}} = \frac{d(\vec{h})}{dt} = w\hat{j}\times\left(-m w r_y r_z\hat{k} + m w r_z^2\hat{j}\right) = -m w^2 r_y r_z\hat{i}$$

In Total :

$$\sum \vec{M}_{/G} = \vec{M}_{A/G} + \vec{M}_{B/G} = \boxed{-2 m w^2 r_y r_z \hat{i}}$$

Conclusion: in order for this rigid body to rotate with a small roll angle , it is required to provide a torque along x axis.

Otherwise, if this torque is absent, the small roll angle cannot be maintained, and the mass A and B start to rotate around the positive x axis. In other words, it will “flip”. This is the phenomenon of Dzhanibekov effect.

The interesting point is that: the effect seems to violate the Newton’s law. Yet it is the Newton’s law that governs this behavior.